



# Historical Lagrangian Dynamics

Marc Lachieze-Rey

► To cite this version:

| Marc Lachieze-Rey. Historical Lagrangian Dynamics. 2014. hal-01081813

**HAL Id: hal-01081813**

**<https://hal.science/hal-01081813>**

Submitted on 11 Nov 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Lagrangian Dynamics of Histories

M. Lachièze-Rey,  
APC - Astroparticule et Cosmologie (UMR 7164)  
Université Paris 7 Denis Diderot 10, rue Alice Domon et Léonie Duquet  
F-75205 Paris Cedex 13, France  
mlr@apc.univ-paris7.fr

November 11, 2014

## Abstract

This paper presents (in its Lagrangian version) a very general “historical” formalism for dynamical systems, including time-dynamics and field theories. It is based on the universal notion of history. Its condensed and universal formulation provides a synthesis and a generalization different approaches of dynamics. It is in our sense closer to its real essence.

The formalism is by construction explicitly covariant and does not require the introduction of time, or of a time function in relativistic theories. It considers space-time (in field theories) exactly in the same manner than time in usual dynamics, with the only difference that it has 4 dimensions. Both time and space-time are considered as particular cases of the general notion of an *evolution domain*.

In addition, the formalism encompasses the cases where histories are not functions (e.g., of time or of space-time), but forms. This applies to electromagnetism and to first order general relativity (that we treat explicitly). It has both Lagrangian and Hamiltonian versions. An interesting result is the existence of a covariant *generalized symplectic form*, which generalizes the usual symplectic or the multisymplectic form, and the symplectic currents. Its conservation on shell provides a genuine symplectic form on the space of solutions.

## 1 Introduction

This paper presents a general formulation of Lagrangian Dynamics, based on the notion of *history*. An history (for “kinematical history”) is a *possible* evolution of a dynamical system. An history which obeys the dynamical equations is a *physical* evolution (or particular solution, or dynamical history).

Dynamics is usually formulated in terms of *dynamical variables* belonging to a *configuration space*. Here we formulate it in terms of histories, i.e., possible evolutions of these variables. Our calculations are thus performed in the *space of histories*. It is infinite dimensional and one achievement is precisely to define such calculus in such an infinite dimensional space.

This formalism describes field theories (hereafter, FT’s) exactly in the same manner than the (usual) time dynamics (hereafter, tD), which appears as a particular case. In time dynamics, an history is a function of time (assumed to

be a well defined notion). Most often, field theories (including the canonical approach to general relativity) are also expressed w.r.t. time (or a *time function*), although the price to pay is to loose covariance. Our approach respects covariance in considering that a field evolve not w.r.t. time, but w.r.t. space-time. Thus, field theories remain entirely covariant and find a very concise expression. It is analog to time evolution, with the one-dimensional time replaced by 4-dimensional space-time (this can be of course generalized to any number of dimensions). The difference does not appear in the formalism, which is designed in that purpose. Evolution is described without any notion of time, or any splitting of space-time.

Moreover, it includes the case where the histories (the fields) are not *functions* like for a scalar field, but *r-forms* on space-time. This applies to the Faraday one-form in electromagnetism, or to the cotetrad and connection forms in first order general relativity.

We define a differential calculus in the infinite-dimensional space of *historical maps*, defined as maps from the space of histories to itself. This allows us to perform a variational calculus in that space, and to obtain an universal dynamical (EL) equation, with a very simple expression. It applies, in an entirely covariant manner, to any field theories and usual equations are recovered as particular cases. We show (in the non-degenerate case) the existence of a canonical *generalized symplectic form* (it reduces to the usual symplectic form in time dynamics), which is covariant and conserved on shell. To mention some general ideas underlying this approach,

- dynamics is not defined versus time, but versus an *evolution domain* which generalizes the time line of tD. This is space-time for FT's, where a space + time splitting is *not* required, and where evolution is described without any time-like parameter.
- An history is not necessarily a function, but is generally defined as a [differential] form on the evolution domain (or, more generally, as a section of some fiber bundle based on it). This includes the case of scalar fields, where functions are seen as zero-forms.
- A particular solution is an history which is an orbit of a dynamical flow in the corresponding bundle. This flow is not one-dimensional like in time dynamics, but has the dimension of the evolution domain. It may be called the *general solution*.
- Our formulation holds in the space of histories which has infinite dimension. Although this is not a manifold, we develop differential calculus in it, in the spirit of *diffeology*[11]. This work may be seen as a formalization and generalization of [20]. The corresponding Hamiltonian approach (not presented here, see [14]) leads to a synthesis between that work and the multisymplectic formalism.
- We define, in the history space, a canonical and covariant generalized symplectic form. This is a generalization of the usual symplectic form in tD, of the multisymplectic form and of the symplectic currents [20] in FTs. We show that it is conserved on shell.

Our formalism (with its Hamiltonian counterpart, [14]) offers a generalized synthesis between the multisymplectic geometry (see, e.g., [10]), the “covariant phase space” approaches (see, e.g., [10]), the canonical approach and the geometry of the space of solutions. It remains entirely covariant.

The section 1 introduces the notion of histories 1.1, in its general sense. It defines their lifts (in the first jet bundle) to *velocity-histories* involved in the Lagrangian dynamics. Section 2 introduces Dynamics in its Lagrangian version: the Euler-Lagrange equation leads to an universal historical evolution equation. We derive the historical expression of Noether theorem. Section 3 applies to electromagnetism and to first order general relativity.

## 1.1 Histories

The central concept is that of *history* (or possible motion, or kinematical, or bare history). According to [17], histories “furnish the raw material from which reality is constructed”.

As a general definition, **an history is an r-form on an evolution domain  $\mathcal{D}$** , and taking its values in a *configuration space*  $Q$  which represents the degrees of freedom of the system: it is a  $Q$ -valued form; a section of a *configuration bundle*  $\mathbf{Q} \rightarrow \mathcal{D}$  which fiber  $Q^{-1}$ . An history (a field) may have components, in which it can be expanded. Each such component is a scalar r-form over  $\mathcal{D}$ , that we will write  $c$ . Without loss of generality, we treat the case of one history components (i.e. scalar-valued r-histories) to which we refer now as “histories”, and that we write  $c$ . This corresponds to the case where  $Q = \mathbb{R}$ . An example of the general case is treated in Section 3.

The *space of histories*  $\mathcal{S} = \text{Sect}(\mathbf{Q})$ , or a subspace of it (for mathematical conditions imposed on these maps, see, e.g., [1]). The space of physical motions (or particular solutions) is a subspace of  $\mathcal{S}$ : those which obey the motion equations, and it is an important task of dynamics to select them<sup>2</sup>.

In the usual time dynamics (tD), the evolution domain is the time line  $\simeq \mathbb{R}$  (or an interval of it). The non relativistic particle, for instance, has configuration space  $\mathbb{R}^3$  (particle position). Each history (zero-history) component is a *function* over time: a zero-form  $c^i : t \rightarrow c^i(t)$  (usually written  $q^i$ ). In usual FTs,  $\mathcal{D}$  is the Minkowski spacetime  $\mathbb{M}_k$  (or a more general space-time): for a scalar field, an history is a scalar function  $c : \mathbb{M}_k \rightarrow \mathbb{R} : m \rightarrow c(m)$  ( $c$  is usually written  $\varphi$ ). In electromagnetism, an history is a scalar value one-form  $A$  (the potential) on space-time. Very generally, we define  $\mathcal{D}$  as a  $n$ -dimensional manifold, possibly with a given metric. (In tD, the existence of time is equivalent to that of a metric  $dt \otimes dt$  for the evolution domain identified to the timeline). For field theories,  $\mathcal{D}$  is in general a metric space-time but conformal or topological theories involve no metric. In the case of general relativity, the metric is dynamical and  $\mathcal{D}$  is a differentiable manifold without prior metric (see below). Our philosophy is to treat  $\mathcal{D}$  as some kind of “ $n$ -dimensional timeline” w.r.t. which the evolution is expressed.

Thus an history (for history component) is a scalar r-form on  $\mathcal{D}$ , to which

---

<sup>1</sup>An r-history may also be seen as a map from  $P^r(\mathcal{D})$ , the space of r-paths of  $\mathcal{D}$ ; such a point of view is convenient for a diffeological analysis [11].

<sup>2</sup> particular solution may be an equivalence class of such histories

we refer as a r-history:

$$c \in \Omega_D^r \stackrel{def}{=} \text{Sect}[\bigwedge^r T^*\mathcal{D}] \subset \Omega_D,$$

the space of sections of the fiber bundle  $\bigwedge^r T^*\mathcal{D} \subset \bigwedge T^*\mathcal{D}$ , (we note  $\bigwedge T^*\mathcal{D}$  the bundle of forms of all degrees; we include *functions* as the case  $r = 0$ ).

We emphasize the necessary distinction between an history, i.e., a section of the fiber bundle  $\mathbf{Q} \rightarrow \mathcal{D}$ , that we always write  $c$ , and its possible values (elements of  $Q$ ) that we write  $\varphi$  (usually written  $q$  in tD). Working with  $\varphi$  would mean working in the finite-dimensional manifold of the configuration bundle  $\mathbf{Q}$ ; working with  $c$ , as we do here, means working in its space of sections. Despite the fact that this is an infinite dimensional space, we define below some differential calculus on it and this a main idea of this paper. Our treatment is inspired by diffeological considerations [11]. Beyond its compactness and generality, we claim that this is closer to the physical reality.

The philosophy of this paper is to transfer the (differential) calculus of configuration space, or phase space, to  $\mathcal{S}$ , or to other spaces with similar status. This provides the possibility of a synthetic treatment applying both to tD and FT in a covariant way. This offers in fact a broader framework which allows us to handle more general situations.

## 1.2 Coordinates

It will be convenient, as an intermediary step for calculations and possibly as a pedagogical help, to assume a system of [local] coordinates on  $\mathcal{D}$ . They will disappear in our final results written in covariant form. A choice of coordinates in  $\mathcal{D}$  generates adapted [local] coordinates in the various fiber bundles we will consider. In tD, this is already provided by time  $t$ . To treat all cases simultaneously, we refer to these coordinates as  $x^\mu$ . We write Vol the volume form defined from these coordinates (when a metric is present, this is *not* the volume form defined by it), and  $\star$  the corresponding Hodge duality (those are non covariant entities). We also write, as usual,  $\text{Vol}_\mu = e_\mu \lrcorner \text{Vol} = \star(dx^\mu)$ ,

$$\text{Vol}_{\mu\nu} = e_\mu \lrcorner \text{Vol}_\nu, \text{ etc.}$$

In tD,  $\mu$  takes the only value  $t$ ;  $\text{Vol} = dt$ ,  $\star\text{Vol} = \star(dt) = 1$ , and  $\text{Vol}_t \stackrel{def}{=} \partial_t \lrcorner \text{Vol} = 1$ . In FT's defined over 4-dimensional space-time  $M$ . In all the paper, we omit the wedge product sign for forms in  $\mathcal{D}$ , and  $\text{Vol} = \epsilon_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma$  (not covariant). We will use multi-index notations defined in A, so that an r-history

$$c = c_{\alpha_1 \dots \alpha_r} dx^{\alpha_1} \dots dx^{\alpha_r} = c_{\underline{\alpha}} d\underline{\alpha}. \quad (1)$$

## 1.3 Velocity-histories

Any section  $c$  of  $\mathbf{Q}$  may be lifted [16] (or *prolongated*) to the first jet bundle  $J^1\mathbf{Q}$  of  $\mathbf{Q}$ . This gives, for any history  $c$ , its first jet extension  $C \stackrel{def}{=} (c, dc)$  that we call the corresponding *velocity-history*. Here  $d$  is the exterior derivative in  $\mathcal{D}$ ;  $dc = c_{,\mu} dx^\mu$ , with  $c_{,\mu} \stackrel{def}{=} \frac{\partial c}{\partial x^\mu}$ , is a one-form on  $\mathcal{D}$ .<sup>3</sup>

<sup>3</sup> This extends without difficulty to  $k^{\text{th}}$  order jets.

Using coordinates, this becomes  $(c, \dot{c})$  — or  $(q, \dot{q})$  — in tD; the multiplet  $(c, c_{,\mu})$  — or  $(\varphi, \varphi_{,\mu})$  — in scalar field theories. To be completely general, we may use multiindexes and represent  $C = (c, dc)$  as the multiplet  $(c_{\underline{\alpha}}, c_{\underline{\alpha},\mu})$  involving the [skew] components of the forms  $c$  and  $dc$  (see A).

We consider  $C$  as a section of the *configuration-velocity bundle*  $\mathbf{V}$ , which identifies with the first jet bundle  $J^1\mathbf{Q}$  of  $\mathbf{Q}$ <sup>4</sup>. Its bundle manifold is called the evolution space (Souriau).

We call

$$\mathcal{S}_V = \text{Sect}(\mathbf{V}) \subset \Omega_{\mathcal{D}}^r \times \Omega_{\mathcal{D}}^{r+1} \subset \Omega_{\mathcal{D}} \times \Omega_{\mathcal{D}}$$

the space of velocity-histories (technically, an *exterior differential system* [4]). Since  $j_1$  is canonical, there is a one-to-one correspondence between  $\mathcal{S}_V$  and  $\mathcal{S}$ , and  $C$  is nothing but a more explicit way to express  $c$ .

A first idea of this paper is to express the (Lagrangian) dynamics in  $\mathcal{S}_V$  instead of  $\mathbf{V}$ , i.e., in the space of sections (histories) rather than the bundle itself. This implies the replacement of functions by *historical maps*, as they are defined below.

## 2 Dynamics : action and Lagrangian

The *action* is a map  $\mathcal{A}$  associating to any history (velocity-history) a real number. It is expressed as the integral

$$\mathcal{A} : C \rightarrow \mathcal{A}[C] = \int_{\mathcal{D}} \mathcal{L}(C).$$

Here  $\mathcal{L}(C) = \mathcal{L}(c, dc)$ , to be integrated on  $\mathcal{D}$ , is an  $n$ -form on  $\mathcal{D}$ . The map  $\mathcal{L}$  associates, to any velocity-history  $C$ , the  $n$ -form  $\mathcal{L}(C)$  on  $\mathcal{D}$ . We call it the *Lagrangian functional*, a specific case of *historical-maps*, defined below.

Let us make the link with the usual physicist's conception, e.g., for a scalar field (zero-history) with a Lagrangian *scalar function*

$$\ell : \mathbf{V} \rightarrow \mathbb{R} : (\varphi, v_{\mu}) \rightarrow \ell(\varphi, v_{\mu}) \quad (2)$$

on the configuration - velocity manifold  $\mathbf{V}$ .<sup>5</sup> The  $n$ -form  $\mathcal{L}(c, dc)$  is defined through

$$\mathcal{L}(c, dc)(x) = \ell[c(x), c_{,\mu}(x)] \text{ Vol.}$$

This remains valid when an history is an  $r$ -form  $c$ , excepted that  $C$  is then expressed by components  $(c_{\alpha_1 \dots \alpha_r}, c_{\alpha_1 \dots \alpha_r, \mu})$  and

$$\ell : \mathbf{V} \rightarrow \mathbb{R} : (\varphi_{\alpha_1 \dots \alpha_r}, v_{\alpha_1 \dots \alpha_r, \mu}) \rightarrow \ell(\varphi_{\alpha_1 \dots \alpha_r}, v_{\alpha_1 \dots \alpha_r, \mu}).$$

---

<sup>4</sup> Strictly speaking,  $J^1\mathbf{Q}$  is defined as a fiber bundle over  $\mathbf{Q}$ . However it defines naturally a fiber bundle over  $\mathcal{D}$ . A velocity-history may be seen as a section of this bundle, which reduces to the tangent bundle in tD (see, e.g., [8, 9]).

<sup>5</sup>The latter admits coordinates  $(x^{\mu}, \varphi, v_{\mu})$  but covariance requirements imply no explicit dependence on the  $x^{\mu}$ .

## 2.1 Historical maps

We define an *historical map* (Hmap) as a generalization of the notion of *functional*:

We consider first maps  $\Omega_D \rightarrow \Omega_D$ : each such map takes an  $r$ -form over  $\mathcal{D}$  as argument and returns an  $R$ -form over  $\mathcal{D}$  ( $r$  and  $R$  are integer  $\leq n$ ). We generalize still further by allowing *two* arguments, so that we define an *historical map* (Hmap) as a map <sup>6</sup>.

$$F : \mathcal{M} \stackrel{def}{=} (\Omega_D)^2 \rightarrow \Omega_D : (c, \gamma) \rightarrow F(c, \gamma).$$

Occasionally we will call such a Hmap a Hform of type  $[0, R]$ : the 0 refers to the fact that this is a map, that we consider as a zero-form (Hforms of higher degree will be considered later); and  $R$  refers to the grade of the values taken by  $F$ .

The wedge product in  $\mathcal{D}$  defines a product of the Hmaps (that, again, we always write implicitly by simple juxtaposition):

$$(F \ G)(c, \gamma) \stackrel{def}{=} F(c, \gamma) \ G(c, \gamma).$$

Thus the Hmaps form an algebra  $\mathcal{F} = \Omega^0(\mathcal{M})$ , that we will treat like an algebra of functions over  $\mathcal{M}$ , that we treat itself as an infinite dimensional manifold. We intend to define differential calculus in that space, allowing variational calculus. This may be seen as a generalization of the variational bicomplex of [2], or of the double complex structure introduced by [6], with the difference that we work in a space of sections rather than in a fiber bundle. Note also that similar approaches ([21], [6]) consider elements of  $\Omega(\text{Sect}(\Omega_D \times D))$ .

In practice we will only consider maps from a certain subset of  $(\Omega_D)^2$ , but we consider here the general case. The Lagrangian functional  $\mathcal{L}$  above is a typical  $[0, n]$ -Hmap. We will however only consider as arguments a pair  $c$  (an  $r$ -history) and  $\gamma = dc$ , the latter being a  $(r+1)$ -form representing its exterior derivative. It returns an  $n$ -form.

First we notice that the differential  $d$  on  $\mathcal{D}$  is easily lifted to  $\mathcal{F}$  through the formula

$$(dF)(c, \gamma) \stackrel{def}{=} d(F(c, \gamma)).$$

It improves the grade from  $[0, R]$  to  $[0, R+1]$ . We call it occasionally “horizontal derivative” but we do not consider it as a genuine part of our differential calculus on  $\mathcal{F}$  since it does not change the status of  $F$  and does not allow variational calculus. In that purpose, we introduce an second “vertical” external exterior derivative  $D$ , different from  $d$  and commuting with it. [12]

## 2.2 Differential calculus for historical maps

We first define the two basic *partial derivative operators*  $\partial_c = \frac{\partial}{\partial c}$  and  $\partial_\gamma = \frac{\partial}{\partial \gamma}$  acting on  $\mathcal{F}$  through the variation formula (wedge product in  $\mathcal{D}$  is always assumed)

$$\delta F = \delta c \frac{\partial F}{\partial c} + \delta \gamma \frac{\partial F}{\partial \gamma}. \quad (3)$$

---

<sup>6</sup> For a  $k$ -order theories,  $(\Omega_D)^2$  would be generalized to  $(\Omega_D)^k$ .

All quantities involved are Hmaps and their arguments  $c$  and  $\gamma$  play the role of “coordinates” in  $\mathcal{M}$ <sup>7</sup>. The types of  $F$ ,  $\frac{\partial F}{\partial c}$  and  $\frac{\partial F}{\partial \text{dc}}$  are respectively  $[0, R]$ ,  $[0, R-r]$  and  $[0, R-r-1]$ . This gives a complete definition, but we give more explicit expressions in Appendix.

We consider  $\frac{\partial}{\partial c}$  and  $\frac{\partial}{\partial \gamma}$  as basic vector-fields on  $\mathcal{M}$ , and define the *general vector-field* as

$$V = V^c \frac{\partial}{\partial c} + V^\gamma \frac{\partial}{\partial \gamma}, \quad (4)$$

whose components  $V^c$  and  $V^\gamma$  are arbitrary Hmaps themselves (wedge product in  $\mathcal{D}$  still assumed). It acts on an arbitrary Hmap  $\beta$ , as

$$V(\beta) = V^c \frac{\partial \beta}{\partial c} + V^\gamma \frac{\partial \beta}{\partial \gamma}.$$

We write  $\chi(\mathcal{M})$  for their set. The generalized multi-vector-field is defined through antisymmetric tensor product.

We now define differential forms: first the *basis one-forms*  $\text{D}c$  and  $\text{D}\gamma$  through their actions on vector-fields:

$$\langle \text{D}c, V \rangle = V^c; \quad \langle \text{D}\gamma, V \rangle = V^\gamma.$$

And the general one-form is  $\alpha = \alpha_c \text{D}c + \alpha_\gamma \text{D}\gamma$ , with components  $\alpha_c$  and  $\alpha_\gamma$  arbitrary Hmaps again. We write  $\Omega^1(\mathcal{M})$  for their set. We may now define the (vertical) exterior derivative of an arbitrary Hmap  $F$  as

$$\text{D}F = \text{D}c \frac{\partial F}{\partial c} + \text{D}\gamma \frac{\partial F}{\partial \gamma}, \quad (5)$$

which is now of type  $[1, R]$ .

The wedge product of forms,  $\wedge$  (not to be confused with the wedge product on  $\mathcal{D}$  written by simple juxtaposition), is the antisymmetrized tensor product, as usual. This defines  $\Omega(\mathcal{M})$ . The table illustrates the action of the horizontal and vertical derivatives  $\text{d}$  and  $\text{D}$ .

$$\begin{array}{ccc} \Omega^0(M) : & [0; R] & \xrightarrow{\text{d}} [0; R+1] \\ & \downarrow \text{D} & \downarrow \text{D} \\ \Omega^1(M) : & [1; R] & \xrightarrow{\text{d}} [1; R+1] \end{array}$$

Since we are interested in variational calculus, we express an arbitrary variation as resulting from the action of a vector-field  $\delta = \delta^c \frac{\partial}{\partial c} + \delta^\gamma \frac{\partial}{\partial \gamma}$  on the arguments  $c$  and  $\gamma$ , namely

$$\delta c = \langle \text{D}c, \delta \rangle = \delta^c, \quad \delta \gamma = \langle \text{D}\gamma, \delta \rangle = \delta^\gamma. \quad (6)$$

This requires  $\delta^c$  and  $\delta^\gamma$  to be of the same grades than  $c$  and  $\gamma$  respectively. It is easy to check that, with 5, this leads to 3. In Lagrangian calculus, we only consider the case where  $\gamma = \text{dc}$ , and thus we restrict to variations obeying  $\delta \gamma = \delta(\text{dc}) = \text{d}(\delta c)$ .

---

<sup>7</sup>  $c$  and  $\gamma$  may be themselves seen as particular (tautological) Hmaps returning their own values.



### 2.3 Historical momentum

Applying this calculus to the Lagrangian gives

$$D\mathcal{L} = Dc \frac{\partial \mathcal{L}}{\partial c} + D(\gamma) \frac{\partial \mathcal{L}}{\partial \gamma}. \quad (7)$$

We call *historical momentum* the Hmap  $P \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial \gamma}$ . It is of type  $[0, n-1]$ , with arguments  $c$  and  $\gamma$ . In all calculations,  $\gamma$  will always take the value  $dc$ . The Hmap  $P$  expands naturally in *dual components*, as  $P = P^\mu \text{Vol}_\mu$ , where we use the polyindexes defined in A. This is defined through  $P^\mu(c, \gamma) = P(c, \gamma)^\mu$ . In time dynamics,  $P$  identifies with the usual momentum. For a scalar field, the dual components  $P^\mu$  correspond to the so called *polymomenta* used in the multisymplectic formalism. The cases of electromagnetism and general relativity are treated below. Similarly, one may call  $\frac{\partial \mathcal{L}}{\partial c}$  the force.

### 2.4 Euler-Lagrange equation

Lagrangian historical dynamics corresponds to the case where  $\gamma = dc$ . Then, since  $d$  and  $D$  commute, the previous identity takes the form

$$D\mathcal{L} = Dc \left( \frac{\delta^{EL} \mathcal{L}}{\delta c} \right) - d\Theta. \quad (8)$$

where we defined the Euler-Lagrange derivative

$$\frac{\delta^{EL} \cdot}{\delta c} \stackrel{def}{=} \frac{\partial \cdot}{\partial c} - (-1)^{|c|} d \frac{\partial \cdot}{\partial (dc)}, \quad (9)$$

with  $|c| = \text{grade of } c$ . We also define the  $[1; n-1]$ -Hform (implicit wedge product in  $\mathcal{D}$ )<sup>8</sup>

$$\Theta \stackrel{def}{=} -Dc P = Dc \frac{\partial \mathcal{L}}{\partial (dc)}.$$

It gives by vertical derivation the  $[2; n-1]$ -Hform  $\omega \stackrel{def}{=} D\Theta = DP \wedge Dc$  (implicit wedge product in  $\mathcal{D}$ ). This is a closed two-form. When the Lagrangian is non singular, it is non degenerate and plays the role of symplectic form, excepted that it take its values in  $\Omega^{n-1}$  instead of  $\mathbb{R}$ . We call  $\Theta$  and  $\omega = D\Theta$  the *historical Lagrangian forms*; in the non-degenerate case, the historical symplectic potential and form. This is the *historical version* of the usual Lagrangian (or symplectic) forms on the velocity-configuration space (see, e.g.[13, 3]).<sup>9</sup>

An arbitrary variation of an history is seen as the action of a vector-field  $\delta$  in  $\mathcal{F}$  as given by 6, such that  $\delta^{dc} = d\delta^c$ . This implies

$$\delta \mathcal{L} = \delta c \left( \frac{\delta^{EL} \mathcal{L}}{\delta c} \right) + d(\delta c \frac{\partial \mathcal{L}}{\partial (d\varphi)}).$$

Since the last term does not contribute to the action, stationarity corresponds to the Euler-Lagrange equation

$$\frac{\delta^{EL} \mathcal{L}}{\delta c} = 0. \quad (10)$$

<sup>8</sup> in the sense of a 1-form for the vertical calculus, and a  $(n-1)$ -form for the [horizontal] calculus in  $\mathcal{D}$ ; see Appendix for notation.

<sup>9</sup> or of the pre-symplectic structure of the evolution space.

This equation applies equally well to time dynamics and field theories. In the latter case, it is explicitly covariant. It includes the case where  $c$  is a r-history (a r-form) rather than a function, as illustrated in the last section.

#### 2.4.1 On shell conservation

On shell, 8 implies

$$D\mathcal{L} = -d\Theta,$$

and thus

$$DD\mathcal{L} = 0 = -Dd\Theta = -dD\Theta = -d\omega :$$

the generalized symplectic form is conserved on shell. This is the covariant version of the conservation of the symplectic current.

Since the value of  $\omega$  is a  $(n-1)$ -form on  $\mathcal{D}$ , it can be integrated along a  $(n-1)$ -dimensional submanifold of  $\mathcal{D}$ . This provides a scalar-valued symplectic form on the space of solutions. On shell, the conservation of  $\omega$  implies that this scalar form does not depend on the choice of the hypersurface (assumed time-like for FTs). This provides the canonical (scalar valued) symplectic form on the space of solutions introduced by [20], so that our result may be seen as a generalization of their work.

### 2.5 Symmetries

A vector-field  $\delta$  is a symmetry generator when it does not modifies the action. This means that it modifies  $\mathcal{L}$  by an exact form  $dX$  only. Hence, for a symmetry,

$$\delta c \left( \frac{\delta^{EL}\mathcal{L}}{\delta c} \right) - d(\delta c P) = dX.$$

Defining the *Noether current* (three-form)  $j \stackrel{def}{=} X + \delta c P$ , we have the conservation law

$$dj = \frac{\delta^{EL}\mathcal{L}}{\delta c} \simeq 0 \text{ (on shell).}$$

Locally,  $j = dQ$ , which defines the *Noether charge density* (n-2)-form  $Q$  [18].

A diffeomorphism of  $\mathcal{D}$  is obviously a symmetry since in that case  $\delta\mathcal{L} = L_\zeta\mathcal{L} = d(\zeta \lrcorner \mathcal{L})$ , where  $\zeta$  is the generator.

## 3 Examples

### 3.1 Time dynamics

An history  $c$  is a function of time: a zero-form on  $\mathcal{D}$  and  $dc = \dot{c} dt$  is a 1-form. Writing  $\ell$  the usual scalar Lagrangian function,

$$\mathcal{L}(c, dc) = L(c, \dot{c}) dt = \ell[c(t), \dot{c}(t)] dt$$

is a one-form. Thus

$$\frac{\partial\mathcal{L}}{\partial c} = \frac{\partial L}{\partial c} dt = \left( \frac{\partial\ell}{\partial q} \circ C \right) dt$$

and

$$\frac{\partial\mathcal{L}}{\partial dc} = \frac{\partial L}{\partial \dot{c}} = \frac{\partial\ell}{\partial v} \circ C$$

are respectively a 1-form and a 0-form.

The EL equations 9 take the usual form:

$$\left(\frac{\partial \ell}{\partial q} \circ C\right) dt + d\left(\frac{\partial \ell}{\partial v} \circ C\right) = \left[\frac{\partial \ell}{\partial q} \circ C + \frac{d}{dt}\left(\frac{\partial \ell}{\partial v} \circ C\right)\right] dt = 0,$$

that physicists usually condense as  $\frac{\partial \ell}{\partial q} + \frac{d}{dt}\left(\frac{\partial \ell}{\partial v}\right) = 0$ .

Note that  $P = P^t$  is a zero-form,  $\text{Vol}_\mu = \text{Vol}_t = 1$ . Then,  $\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial c} \delta c + P \delta \dot{c} dt = \frac{\partial \mathcal{L}}{\partial c} \delta c + P \delta(\text{dc})$ : a covariant expression of  $\text{tD}$ .

### 3.2 Electromagnetism in Minkowski spacetime

The dynamical variable is the one-form (1-history)  $A$ . The Lagrangian functional  $\mathcal{L} = \frac{1}{2} dA (\star dA)$  (Hodge duality in Minkowski spacetime). It results  $\frac{\partial \mathcal{L}}{\partial(dA)} = \star dA$  and the Euler-Lagrange equation ,

$$d(\star dA) = 0 \Leftrightarrow \square A = 0,$$

reduces to the Maxwell equation.

### 3.3 First order gravity

We start from the Lagrangian functional (on a 4-dimensional differentiable manifold  $M$  without metric)

$$\mathcal{L} = \epsilon_{IJKL} e^I e^J (d \omega^{KL} + (\omega \omega)^{KL}). \quad (11)$$

(Again, we forget the wedge product signs with the convention that juxtaposed forms on  $M$  are wedge-multiplied). The dynamical variables are the cotetrad one-forms  $e^I$  and the Lorentz-connection one-forms  $\omega^{KL}$ , with respective momenta  $P_I$  and  $\Pi_{KL}$ . We have

$$\mathcal{L}_I \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial e^I} = 2 \epsilon_{IJKL} e^J (d \omega^{KL} + (\omega \omega)^{KL}) \quad \text{and} \quad P_I \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \text{de}^I} = 0; \quad (12)$$

$$\mathcal{L}_{KL} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \omega^{KL}} = 2 \epsilon_{IJNL} e^I e^J \omega_K^N = 2 \epsilon_{IJKL} e^I \omega_M^J e^M \quad (13)$$

(after some algebra) and

$$\Pi_{KL} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial d\omega^{KL}} = \epsilon_{IJKL} e^I e^J. \quad (14)$$

The EL equations are easily derived as

$$\frac{\partial \mathcal{L}}{\partial e^I} = 2 \epsilon_{IJKL} e^J (d \omega^{KL} + (\omega \omega)^{KL}) = 0$$

which means zero Ricci curvature; and

$$\frac{\partial \mathcal{L}}{\partial \omega^{KL}} + d \frac{\partial \mathcal{L}}{\partial d\omega^{KL}} = \epsilon_{IJKL} e^I [\omega_M^J e^M + \text{de}^J] = 0$$

which means zero torsion.

### 3.4 Lorentz invariance

An infinitesimal Lorentz transformation transforms the tetrad and the connection as

$$\begin{aligned}\delta e^I &= \lambda^I{}_J e^J; \\ \delta \omega^{IJ} &= \lambda^I{}_K \omega^{KJ} - \omega^{IK} \lambda_K^J + d\lambda^{IJ}.\end{aligned}$$

The Lorentz invariance of  $\mathcal{L}$  leads to the conserved Noether current: the 3-form (on  $\mathcal{D}$ )

$$J = \Pi_{IJ} \delta \omega^{IJ} = \Pi_{IJ} (\lambda^I{}_K \omega^{KJ} - \omega^{IK} \lambda_K^J + d\lambda^{IJ}).$$

Conservation takes the form

$$0 = dJ = d\Pi_{IJ} (\lambda^I{}_K \omega^{KJ} - \omega^{IK} \lambda_K^J + d\lambda^{IJ}) + \Pi_{IJ} d(\lambda^I{}_K \omega^{KJ} - \omega^{IK} \lambda_K^J).$$

This splits as

$$\begin{aligned}0 &= d\Pi_{IJ} (\lambda^I{}_K \omega^{KJ} - \omega^{IK} \lambda_K^J) + \Pi_{IJ} (\lambda^I{}_K d\omega^{KJ} - d\omega^{IK} \lambda_K^J); \\ 0 &= d\Pi_{IJ} (d\lambda^{IJ}) + \Pi_{IJ} (d\lambda^I{}_K \omega^{KJ} - \omega^{IK} d\lambda_K^J).\end{aligned}$$

The first expresses the *global* invariance as

$$\begin{aligned}0 &= d\Pi_{IJ} \lambda^I{}_K \omega^{KJ} - d\Pi_{IJ} \omega^{IK} \lambda_K^J + \Pi_{IJ} \lambda^I{}_K d\omega^{KJ} - \Pi_{IJ} d\omega^{IK} \lambda_K^J \\ &= 2 \lambda_K^A d(\Pi_{AJ} \omega^{KJ}).\end{aligned}$$

Since this is for arbitrary  $\lambda$ , and taking account of antisymmetry, this implies (on shell)

$$d\mathcal{J}_{AB} = 0; \quad \mathcal{J}_{AB} = \Pi_{[AJ} \omega^J{}_{B]} = \epsilon_{[AJMN} e^{MN} \omega^J{}_{B]}.$$

Algebraic calculations show that this is a direct consequence of the motion equations.

#### Local invariance

$$0 = d\Pi_{IA} (d\lambda^{IA}) + \Pi_{IJ} (d\lambda_K^J \omega^{IK} - d\lambda^I{}_K \omega^{KJ});$$

$$0 = (d\Pi_{AB} - \Pi_{JB} \omega^J{}_A + \Pi_{AJ} \omega_B^J) d\lambda^{AB}.$$

This implies, taking antisymmetry into account,

$$d\Pi_{AB} = \Pi_{JB} \omega^J{}_A - \Pi_{AJ} \omega_B^J = 0,$$

also a consequence of the motion equations.

## 4 Conclusions

We have expressed dynamics in terms of histories rather than dynamical variables. Although the set of histories has infinite dimensions, we have defined a differential calculus on it which allowed us to perform variational calculus. A first benefit is that this formulation remains entirely covariant and does not require, for relativistic field theories (including general relativity), the introduction of a time variable. In addition, it applies to the case where the histories

(the “ fields”) are forms rather than functions, like for electromagnetism or the first order formulations of general relativity. Such forms are treated as genuine objects and not decomposed into their components.

We have derived a very simple expression for the dynamical equations, analog to the usual *Lagrange equations*, but with an extended generality (covariance and treatment of forms). We have derived (generalizations of) the Lagrangian, presymplectic and symplectic forms, and shown the conservation of the latter on shell. We have also defined the conserved currents associated to symmetries.

All these quantities, as well as (the generalization of ) observables are well defined, although they are not usual functions or forms, but rather new mathematical entities that we have called Hmaps, for which we have defined proper calculations. Although an *observable* is generally considered as a scalar-valued function, it appears here (in particular in the case of general relativity) as a form-valued map. We consider this generalization as a natural benefit of our approach since any form provides a scalar by integration over a submanifold of adapted dimension. This provides a procedure to extract scalar observables from the generalized (form-valued) observables, with the help of intermediary submanifolds. This corresponds for instance to what is done in Loop Quantum Gravity through the introduction of the Holonomy-Flux algebra.

The observables, as they are naturally defined here, depend on histories, not on the configuration or phase space variables.<sup>10</sup> Since, by construction, they depend on the whole history, they remain constant during the evolution. For instance, they necessarily commute with the Hamiltonian and with the constraints, so that they correspond to the *complete observables* in the sense of [15, 7] (see also[19]).

In a following paper [14], one defines a generalized Legendre transform and an explicetly covariant Hamiltonian historical dynamics.

## Appendices

### A Historical Functionals and Multi-index notations

We introduce a multi-index notation for skew indices characterizing the components of forms : we write  $\underline{\alpha}$  for the antisymmetrized sequence  $\alpha_1 \dots \alpha_r$ . In addition, we write  $d^{\underline{\alpha}} = dx^{\alpha_1} \dots dx^{\alpha_r}$  and the multivector  $e_{\underline{\alpha}} = (e_{\alpha_1}, \dots, \alpha_r)$  (wedge products assumed).

An r-forms expands as

$$c = c_{\underline{\mu}} d^{\underline{\mu}} = c_{\mu_1 \dots \mu_r} dx^{\mu_1} \dots dx^{\mu_r}.$$

The notation extends to Hmaps through

$$F = F_{\underline{\alpha}} d^{\underline{\alpha}} : \text{as } F_{\underline{\alpha}}(c, \gamma) = (F(c, \gamma))_{\underline{\alpha}}.$$

---

<sup>10</sup>In the Hamiltonian formalism [14], It is possible reformulate them as functions, e.g., on phase space

Then we have

$$\frac{\partial F}{\partial c} = \frac{\partial F_{\underline{\alpha}}}{\partial c_{\underline{\mu}}} (e_{\underline{\mu}} \lrcorner d^{\underline{\alpha}}). \quad (15)$$

In the case where a form  $P$  is expanded in dual components:  $P = P^{\underline{\mu}} \text{Vol}_{\underline{\mu}}$ , with  $\text{Vol}_{\underline{\mu}} = e_{\underline{\mu}} \lrcorner \text{Vol}$ , then

$$\frac{\partial F}{\partial P} = \epsilon^{\underline{\mu}\underline{\nu}} \frac{\partial F_{\underline{\alpha}}}{\partial P^{\underline{\mu}}} (e_{\underline{\nu}} \lrcorner d^{\underline{\alpha}}), \quad (16)$$

where  $\epsilon^{\underline{\mu}\underline{\nu}} \stackrel{\text{def}}{=} \epsilon^{\mu_1 \dots \mu_r \nu_1 \dots \nu_{n-r}}$ .

The validity of these formulas implies some conditions on the grades of the forms involved, that we always assume fulfilled.

In the Lagrangian case,  $\mathcal{L} = \ell \text{Vol}$ ,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= \frac{\partial \ell}{\partial c_{\underline{\mu}}} \text{Vol}_{\underline{\mu}}; & \frac{\partial \mathcal{L}}{\partial (dc)} &= \frac{\partial \ell}{\partial c_{\underline{\mu}, \nu}} \text{Vol}_{\underline{\mu}, \nu}. \\ \text{D}\mathcal{L} &= \text{D}c \frac{\partial \mathcal{L}}{\partial c} + \text{D}(dc) \frac{\partial \mathcal{L}}{\partial dc}, \end{aligned} \quad (17)$$

where antisymmetrization over all indices is assumed.

## References

- [1] J. Fernando Barbero G., Jorge Prieto, Eduardo J. S. Villaseor, *Hamiltonian treatment of linear field theories in the presence of boundaries: a geometric approach*, <http://arxiv.org/abs/1306.5854>
- [2] THOMAS J. BRIDGES, PETER E. HYDON and JEFFREY K. LAWSON (2010). Multisymplectic structures and the variational bicomplex. *Mathematical Proceedings of the Cambridge Philosophical Society*, 148, pp 159-178. doi:10.1017/S0305004109990259.
- [3] Semiclassical quantization of classical field theories. A. S. Cattaneo, P. Mnev and N Reshetikhin. <http://arxiv.org/abs/1311.2490v1>
- [4] H. Cendra and S. Capriotti, *Cartan Algorithm and Dirac constraints for Griffiths Variational Problems*, <http://arxiv.org/abs/1309.4080v1>
- [5] Methods of Differential Geometry in Classical Field Theories: k-symplectic and k-cosymplectic approaches M. de León, M. Salgado & S. Vilariño, <http://arxiv.org/abs/1409.5604v1>
- [6] Quantum fields and strings, Deligne & Freed
- [7] B. Dittrich, *Partial and Complete Observables for Hamiltonian Constrained Systems*, <http://arxiv.org/abs/gr-qc/0411013v1>
- [8] M. J. Gotay 1991, *A multisymplectic framework for classical field theory and the calculus of variations I. Covariant Hamiltonian formalism*, *Mechanics, Analysis, and Geometry: 200 Years After Lagrange* (M. Francaviglia, ed.), 203235, North Holland, Amsterdam.

- [9] Mark J. Gotay, James Isenberg, Jerrold E. Marsden, Richard Montgomery, *Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory*, <http://arxiv.org/abs/physics/9801019>
- [10] Frédéric Hélein, *Multisymplectic formalism and the covariant phase*, <http://arxiv.org/abs/1106.2086>
- [11] Patrick Iglesias-Zemmour, *Diffeology*, American Mathematical Soc., 2013
- [12] Igor Khavkine, Covariant phase space, constraints, gauge and the Peierls formula, <http://arxiv.org/abs/1402.1282v2>
- [13] Some applications of differential geometry in the theory of mechanical systems M.P. Kharlamov, <http://arxiv.org/abs/1401.8233>
- [14] Marc Lachièze-Rey, Historical Hamiltonian formalism, in preparation
- [15] Carlo Rovelli, *Partial observables*, <http://arxiv.org/abs/gr-qc/0110035>
- [16] W. Sarleta and G. Waeyaert, *Lifting geometric objects to the dual of the first jet bundle of a bundle fibred over R*, <http://arxiv.org/abs/1308.3086v1>
- [17] Rafael D. Sorkin, *Logic is to the quantum as geometry is to gravity*, <http://arxiv.org/abs/1004.1226>
- [18] Robert M. Wald, *Black Hole Entropy is Noether Charge*, <http://arxiv.org/abs/gr-qc/9307038v1>
- [19] Hans Westman and Sebastiano Sonego, *Events and Observables in generally invariant Spacetime Theories*, <http://arxiv.org/abs/0708.1825v2>
- [20] Cedomir Crnkovic and Edward Witten, *Covariant description of canonical formalism theories in geometrical theories*, in *Three hundred Years of Gravitation*, eds. S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987) pp. 676–684.
- [21] G. J. Zuckerman, *Action principles and global geometry, Mathematical aspects of string theory*, ed. S. T. Yau, World Scientific Publishing, 1987.